Fall 2015

8. $\hat{u}(w,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(x,t)e^{iwx}dx$

$$\hat{u}_t + 7(-iw)\hat{u} = 0$$

$$\hat{u}(w,t) = C(w)\cdot e^{7iwt}$$

$$\hat{u}(w,0) = \hat{f}(w) = C(w)$$

$$\hat{u}(w,t) = \hat{f}(w)\cdot e^{7iwt}$$

$$u(x,t) = \int_{-\infty}^{+\infty} \hat{f}(w)\cdot e^{7iwt}\cdot e^{-iwx}dw$$

$$= \int_{-\infty}^{+\infty} \hat{f}(w)\cdot e^{-iw(x-7t)}dw$$

$$= f(x-7t)$$

9. $u(r,\theta,t) = \phi(r,\theta)\cdot G(t)$.

$$\frac{\Delta\phi}{\phi} = -\lambda = -\frac{G''(t)}{G(t)}$$

$$\begin{cases} \Delta\phi + \lambda\phi = 0, \\ \phi(1,\theta) = 0 \end{cases}$$

$$\phi(r,\theta) = R(r)\,V(\theta)$$

$$-\lambda = \frac{\frac{1}{r}(rR')'}{R} + \frac{1}{r^2}\frac{V''(\theta)}{V(\theta)}$$

$$\begin{cases} V''(\theta) = -\mu V(\theta) \\ V(-\pi) = V(\pi) \\ V'(-\pi) = V'(\pi) \end{cases} \Rightarrow \mu = n^2$$

$$V(\theta) = \begin{cases} \sin n\theta \\ \cos n\theta \end{cases}$$

$n = 0, 1, 2, \ldots$

$R(r) = J_n(\sqrt{\lambda}\, r)$.

$R(1) = 0 \Rightarrow \sqrt{\lambda} = z_{nm}$,

$$G(t) = \begin{cases} \cos\sqrt{\lambda}\, t \\ \sin\sqrt{\lambda}\, t . \end{cases} \qquad G'(t) = \begin{cases} -\sqrt{\lambda}\sin\sqrt{\lambda}\, t \\ \sqrt{\lambda}\cos\sqrt{\lambda}\, t . \end{cases}$$

$u(r, \theta, 0) = 0$

$u_t(r, \theta, 0) = \sum \frac{1}{m^2} J_0(z_{0m} r).$

$$u(r, \theta, t) = \sum_{m=1}^{+\infty} \frac{1}{m^2 \cdot z_{0m}} J_0(z_{0m} r) \sin(z_{0m} t)$$

(only $n=0$ terms left, because no $\begin{cases} \cos n\theta \\ \sin n\theta \end{cases}$ $n \geq 1$ in $u(r, \theta, 0)$ $u_t(r, \theta, 0)$)

Fall 2016

8. The same as problem 9 in 2015 Fall Final.

$$u(r,\theta,z) = \sum_{n=1}^{+\infty} \frac{1}{n} J_0(z_{0n} r) \cdot \sinh(z_{0n} z)$$

9. ① Inhomogeneous BCs

$u_0(x)$ satisfies

$$\begin{cases} u_0(0)=0 \\ u_0(1)=3, \\ u''(x)=0 \end{cases} \Rightarrow u_0(x) = 3x.$$

② $w(x,t) = u(x,t) - u_0(x)$

$$\Rightarrow \begin{cases} w_t = w_{xx} + e^{-t}\sin 2\pi x \\ w(0,t)=0 \\ w(1,t)=0 \end{cases}$$

$w(x,0) = x - 3x = -2x.$

Eigenfunction expansion

$$\begin{cases} \phi''(x) = -\lambda \phi(x) \\ \phi(0) = \phi(1) = 0 \end{cases}$$

$\phi_n(x) = \sin n\pi x. \quad \lambda_n = -(n\pi)^2.$

$$w(x,t) = \sum_{n=1}^{+\infty} A_n(t)\, \phi_n(t)$$

$$w_t = w_{xx} + e^{-t} \sin 2\pi x$$

$$\Rightarrow \quad A_n'(t) = -(n\pi)^2 A_n \quad (n \neq 2)$$

$$A_2'(t) = -(2\pi)^2 A_2 + e^{-t}$$

$$A_n(t) = e^{-(n\pi)^2 t} \cdot C_n. \quad n \neq 2$$

$$A_n(0) = \int_0^1 -2x \cdot \sin(n\pi x)$$

$$= \frac{4}{n\pi}\left((-1)^n - 1\right)$$

$$A_n(t) = e^{-(n\pi)^2 t} \cdot \left(\frac{2}{n\pi} \left((-1)^n - 1\right) \right)$$

$$A_2'(t) = \underbrace{C \cdot e^{-t}}_{\downarrow} + C_2 \cdot e^{-(2\pi)^2 t}$$

$$-C \cdot e^{-t} = -(2\pi)^2 C e^{-t} + e^{-t}$$

$$C = \frac{1}{(2\pi)^2 - 1}$$

$$A_2(t) = \frac{1}{(2\pi)^2 - 1} e^{-t} + C_2 \cdot e^{-(2\pi)^2 t}$$

$$C_2 = A_2(0) - \frac{1}{(2\pi)^2 - 1}$$

$$= -\frac{1}{(2\pi)^2 - 1}.$$

$$A_2(t) = \frac{1}{(2\pi)^2 - 1} e^{-t} - \frac{1}{(2\pi)^2 - 1} e^{-(2\pi)^2 t}$$

$$u(x,t) = \sum_{\substack{n=1 \\ n \neq 2}}^{+\infty} \frac{4}{n\pi}\left((-1)^n - 1\right) \sin(n\pi x) \cdot e^{-(n\pi)^2 t}$$

$$+ \sin(2\pi x)\left(\frac{1}{(2\pi)^2 - 1} e^{-t} - \frac{1}{(2\pi)^2 - 1} e^{-(2\pi)^2 t}\right)$$

$$+ 3x.$$

Fall 2013 Problem 7.

Find $u_E(x,t)$.

①. $$\begin{cases} u_{Et} = u_{Exx} + x \\ u_E(0,t) = 0 \\ u_{Ex}(1,t) = t, \end{cases}$$

$u_E = xt$.

② $W(x,t) = u - u_E$.

w satisfies

$$\begin{cases} w_t = w_{xx} + 2e^{-\frac{\pi^2}{4}t} \sin \frac{\pi x}{2} \\ w(0,t) = 0 \\ w_x(1,t) = 0 \\ w(x,0) = 5 \sin \frac{3\pi x}{2} \end{cases}$$

Use eigen function expansion.

$$w(x,t) = \sum_{n=1}^{+\infty} A_n(t) \sin \frac{2n-1}{2}\pi x$$

$$A_n'(t) + \left(\left(\frac{2n-1}{2}\right)\pi\right)^2 A_n(t) = 0 \quad n \neq 1.$$

(Use initial condition to determine $A_n(t)$).

$$A_1'(t) + \left(\frac{\pi}{2}\right)^2 A_1(t) = 2e^{-\frac{\pi^2}{4}t}.$$

Guess the solution:

$$A_1(t) = C\cdot t e^{-\frac{\pi^2}{4}t} + C_1 e^{-\frac{\pi^2}{4}t}.$$

Then $$C\left(e^{-\frac{\pi^2}{4}t} + t\left(-\frac{\pi^2}{4}\right)e^{-\frac{\pi^2}{4}t} + \left(\frac{\pi^2}{4}\right)\cdot t e^{-\frac{\pi^2}{4}t}\right) = 2e^{-\frac{\pi^2}{4}t}.$$

So $C = 2$ (Use Initial condition to determine C_1)

So the solution $$w(x,t) = 2t\sin\frac{\pi x}{2}e^{-\frac{\pi^2}{4}t} + 5\sin\frac{3\pi}{2}x\, e^{-\frac{9\pi^2}{4}t}$$

$$u(x,t) = w(x,t) + xt.$$